

FAMILIES OF SHORT CYCLES ON RIEMANNIAN SURFACES

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ABSTRACT. Let M be a closed Riemannian surface of genus g . We construct a family of 1-cycles on M that represents a non-trivial element of the k 'th homology group of the space of cycles and such that the mass of each cycle is bounded above by $C \max\{\sqrt{k}, \sqrt{g}\} \sqrt{\text{Area}(M)}$. This result is optimal up to a multiplicative constant.

1. INTRODUCTION

Let M be a closed Riemannian 2-dimensional manifold and let $Z_1(M, \mathbb{Z}_2)$ denote the space of mod 2 flat 1-cycles in M . Let Z_1^0 denote the connected component of $Z_1(M, \mathbb{Z}_2)$ consisting of all null-homologous cycles in M . It follows from the work of Almgren [1] that Z_1^0 is weakly homotopy equivalent to the Eilenberg-MacLane space $K(\mathbb{Z}_2, 1) \simeq \mathbb{RP}^\infty$. We say that a family of cycles $f : \mathbb{RP}^k \rightarrow Z_1^0$ is a k -sweepout if it represents the non-zero element of the k 'th homology group $H_k(Z_1^0, \mathbb{Z}_2) \cong \mathbb{Z}_2$.

Here is the main result of this paper.

Theorem 1.1. *Let M be a 2-dimensional closed Riemannian manifold of genus g . For each k there exists a k -sweepout $\mathcal{Z}_k = \{z_t\}_{t \in \mathbb{RP}^k}$ of M , such that for each $t \in \mathbb{RP}^k$ the mass of z_t is bounded above by $1600 \max\{\sqrt{k}, \sqrt{g}\} \sqrt{\text{Area}(M)}$.*

k -sweepouts have been studied by Gromov in [9], [11] and [12] and by Guth in [14]. More recently, in [23] Marques and Neves used k -sweepouts to prove existence of infinitely many minimal hypersurfaces in manifolds of positive Ricci curvature. In [7] Glynn-Adey and the author obtained upper bounds for volumes of these hypersurfaces.

In the case of surfaces Balacheff and Sabourau [2] constructed a sweepout of M by 1-cycles of mass bounded by $C \sqrt{(g+1) \text{Area}(M)}$. This corresponds to the case $k = 1$ of Theorem 1.1. Different proofs of their result, improving the value of an upper bound for the constant C , were given in [21], [7]. The proof of Balacheff and Sabourau relies on the estimate of Li and Yau [20] for the first eigenvalue of the Laplacian. In this paper we give an elementary construction of k -sweepouts using only the thin-thick decomposition of hyperbolic surfaces and the length-area method.

The upper bound in Theorem 1.1 is optimal up to a constant. Brooks constructed examples of closed hyperbolic surfaces of arbitrarily large genus such that any 1-sweepout of Σ_g must contain a cycle of mass greater than $c\sqrt{g}$ for some $c > 0$. On the other hand, Gromov showed in [9] that a k -sweepout of the round n -sphere by $(n-1)$ -cycles must contain a cycle of mass greater than $ck^{\frac{1}{n}}$ for a constant $c > 0$. To prove this Gromov observed that if $\{U_i\}$ is a collection of k disjoint measurable subsets in M and z_t is a k -sweepout, then there will be a cycle z_t that separates each U_i into two subsets of equal area. Gromov's arguments were generalized and extended by Guth in [15]. In that paper Guth proves nearly optimal lower and upper bounds for all homology classes of the space of mod 2 m -dimensional cycles on the n -dimensional round sphere.

In [9] Gromov suggested that finding bounds on the maximal mass of a cycle in an optimal k -sweepout can be thought of as a non-linear analogue of the spectral problem on M . Arguments in our paper, especially the use of the length-area method, were inspired by and are similar to the estimates for the eigenvalues of the Laplace operator on Riemannian manifolds in the works of Hersch [17], Yau [25], Yang and Yau [24], Korevaar [19], Gromov [10], Grigoryan, Netrusov and Yau [8], Colbois and Maertens [5], and Hassannezhad [16].

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2. OUTLINE OF THE PROOF

Let M be a closed surface of area 1. Suppose we can cover M by k sets U_i with piecewise smooth boundary and disjoint interiors, each of area $\sim \frac{1}{k}$, and such that the boundary length of each set is $\sim \frac{1}{\sqrt{k}}$. Assume furthermore that for each U_i there exists a 1-sweepout of U_i by cycles of length at most $\sim \frac{1}{\sqrt{k}}$. We can now sweep out all of M as follows. First we sweep out U_1 , starting on a 0-cycle and ending on the boundary of U_1 . We hold cycle ∂U_1 fixed and start adding to it a sweepout of U_2 and so on. Eventually cycles in the boundaries of U_i 's will overlap and cancel out.

Denote this sweepout of M by z_t and consider a cycle $z = \sum_{i=1}^k z_{t_i}$, where $\{t_i\}$ are k different moments of time. Each z_{t_i} can be decomposed into two parts: one that

lies in $\bigcup \partial U_i$ and one that is contained in only one of the sets U_i and has mass at most $\sim \frac{1}{\sqrt{k}}$. Since the cycles are mod 2, the parts that overlap in $\bigcup \partial U_i$ will cancel out, so $mass(z) \lesssim \sqrt{k}$. There exists a k -sweepout of M that consists of cycles like z and therefore satisfies the desired upper bound.

The idea described above was successfully used by Gromov and Guth to bound volumes of k -sweepouts in various contexts.

If M is a Riemannian 2-sphere then one can find a covering of M by k sets as described above. This can be done using the length-area method as described in Section 7. To construct a 1-sweepout of U_i 's we use the following idea from the work of Balacheff and Sabourau [2]. First, we find a relative 1-cycle c_1 subdividing U_i into two sets U_i^1 and U_i^2 each of area $\leq r \text{Area}(U_i)$ for some fixed $r \in (0, 1/2)$, such that the length of c_1 is bounded above by $\sim \sqrt{\text{Area}(U_i)}$. Let $W_1(U)$ denote the maximal length of a relative cycle in an "optimal" sweepout of U (precise definition will be given in section 3). Given a sweepout of each of U_i^1 and U_i^2 by relative cycles we can assemble them into a sweepout of U_i by attaching pieces of c_1 to some of these cycles. It follows then that $W_1(U_i)$ is bounded above by $\sim \max\{W(U_i^1), W(U_i^2)\} + \sqrt{\text{Area}(U_i)}$. We can repeat this process and subdivide U_i^j into two subsets $U_i^{j,1}$ and $U_i^{j,2}$. After n iterations we obtain $W_1(U_i) \lesssim \max\{U_i^{j_1, \dots, j_n}\} + \sum_{i=0}^{n-1} r^i \sqrt{\text{Area}(U_i)}$ and the areas of sets $U_i^{j_1, \dots, j_n}$ are at most $r^n \text{Area}(U_i)$. Since the geometric series $\sum_{i=0}^{n-1} r^i$ converges as $n \rightarrow \infty$ the above argument reduces the problem of bounding the 1-width of U_i to a problem of bounding the 1-width of a subset $U_i^{j_1, \dots, j_n} \subset U_i$ of arbitrarily small area. To accomplish this we cut $U_i^{j_1, \dots, j_n}$ into pieces which are $(1+\epsilon)$ -bilipschitz to open subsets of Euclidean plane and apply an argument of Guth [13].

However, if the surface has genus greater than k the above argument may not work. It may happen that every collection of k open sets of approximately equal areas that cover M have large length of the boundary and some of these open sets do not admit a sweepout by short cycles. This happens, for example, for hyperbolic surfaces of high genus constructed by Brooks [3].

Instead we will first cover M by $\sim g$ 'good regions' V_i (where g is the genus). These regions can have arbitrary areas, but they have the following nice properties:

- (1) There exists a sweepout of V_i by relative 1-cycles of length at most $\sim \sqrt{\text{Area}(V_i)}$
- (2) We can subdivide V_i into m (where m is any positive integer) subsets of approximately equal areas, such that the length of the union of their boundaries is at most $\sim \sqrt{m} \sqrt{\text{Area}(V_i)} + l(\partial V_i)$

So for our purposes these good regions are as good as subsets of the sphere. We will then subdivide them into subsets of the right area. The value of m that we choose for each region V_i will depend on k and the area of V_i .

To obtain these good regions we use uniformization theorem and the length-area method. By uniformization theorem a surface of genus $g \geq 2$ is conformally equivalent to a hyperbolic surface. P. Buser used thin-thick decomposition to construct a tessellation of a hyperbolic surface by polygons of approximately equal areas with some special properties. The thin part of the surface in this tessellation is covered by long and narrow rectangles and the thick part is covered by triangles that are close to equilateral triangles. For us the most important thing about this tessellations is that every polygon contains at most c other polygons in its $1/2$ -neighbourhood. Our good regions will be those that are covered by at most c polygons from this tessellation.

To control lengths of the boundaries of good regions we observe that if a family of concentric geodesic circles (i.e. level sets of the distance function) on the hyperbolic surface (conformal to our surface M) covers a set of small area, when measured with the original (non-hyperbolic) metric, then some of these circles must be short in the original metric. This is a classical observation sometimes called the length-area method (see Section 4). We use it to find short cycles on M in $1/2$ -neighbourhood of a polygon from the hyperbolic tessellation. Actually, the length of the boundary of each individual good region in our construction may be comparatively long, but the total length of the union of their boundaries will be at most $\sim \max\{\sqrt{g}, \sqrt{k}\}$. Moreover, after we subdivide each good region into smaller parts using property (2) above so that area of each part is at most $\sim \frac{1}{k}$, the total length of the union of the boundaries of all parts will still be at most $\sim \max\{\sqrt{g}, \sqrt{k}\}$. This is sufficient to bound lengths of k -sweepouts using the argument described above.

Here's the plan of the paper. In Section 3 we define k -sweepouts and a technical notion of monotone sweepouts. These sweepouts have a nice property that it is easy to glue two short monotone sweepouts of adjacent regions into a short monotone sweepout of their union. In Section 4 we use the length-area method to prove a key lemma for finding subsets of M with small length of the boundary. In Section 5 we describe Buser's tessellation \mathcal{T} of a hyperbolic surface by quadrilaterals and triangles. In Section 6 we describe Guth's construction of sweepouts of open subsets of \mathbb{R}^2 . We use this result as the base of induction in the proof that a subset of M of very small area admits a sweepout by short cycles. In Section 7 we prove that if a subset U of M can be covered by at most 40 elements of \mathcal{T} then it admits a sweepout by cycles of length at most $\sim \sqrt{\text{Area}(U)}$. In Section 8 we construct a covering of M by sets that are contained in at most 40 elements of \mathcal{T} and have area at most $\frac{\text{Area}(M)}{k}$ and finish the proof of the theorem.

3. PRELIMINARIES

For the definition of the space of mod 2 cycles with flat metric we refer the reader to [6] or a concise description in [2, Section 2], which will be sufficient for our purposes.

In [1] Almgren constructed maps from homotopy groups of the integral cycle space $\pi_k(Z_m(M^n, \mathbb{Z}); 0)$ to homology groups of the manifold $H_{k+m}(M^n, \mathbb{Z})$ and proved that these maps are isomorphisms for all non-negative integers k and m . Almgren's proof works for \mathbb{Z}_2 coefficients as well. For a surface M we have an isomorphism $\pi_k(Z_1(M, \mathbb{Z}_2); 0) \cong H_{k+1}(M, \mathbb{Z}_2)$. Since homology groups of M are zero for $k > 1$, the connected component Z_1^0 of $Z_1(M, \mathbb{Z}_2)$, $0 \in Z_1^0$, is weakly homotopy equivalent to the Eilenberg-MacLane space $K(\mathbb{Z}_2, 1) \simeq \mathbb{RP}^\infty$.

For a surface M Almgren's map $F_A : \pi_1(Z_1(M, \mathbb{Z}_2), 0) \rightarrow H_2(M, \mathbb{Z}_2)$ is defined as follows. Consider a loop $z_t : S^1 \rightarrow Z_1(M, \mathbb{Z}_2)$ representing some class of the fundamental group and pick a fine subdivision $\{t_1, \dots, t_n\}$ of S^1 . For each t_i cycle z_{t_i} can be approximated by a cycle that consists of a finite collection of Lipschitz circles. If c_i and c_{i+1} are two such approximations of z_{t_i} and $z_{t_{i+1}}$ respectively, we can find an area minimizing chain A_i with $\partial A_i = c_i - c_{i+1}$. We can then assemble chains A_i into a 2-cycle that represents an element of $H_2(M, \mathbb{Z}_2)$. It turns out that if the subdivision and approximations are fine enough then the 2-cycle will represent the same element in the homology independent of the particular subdivision and approximations.

We say that $\{z_t\}_{t \in \mathbb{RP}^1}$ is a sweepout (or 1-sweepout) of M if loop $\{z_t\}$ is non-contractible in $Z_1(M, \mathbb{Z}_2)$, i.e. $F_A([z_t]) \neq 0$. More generally, we say that $\{z_t\}_{t \in \mathbb{RP}^k}$ is a k -sweepout if it represents the non-zero element of $H_k(Z_1^0) \cong \mathbb{Z}_2$. The ring structure of $H^*(Z_1^0, \mathbb{Z}_2) \cong \mathbb{Z}_2[a]$, where a is the non-zero class of $H^1(Z_1^0, \mathbb{Z}_2)$, provides a useful criterion for when a family is a k -sweepout. We have that map $f : \mathbb{RP}^k \rightarrow Z_1^0$ is a k -sweepout if and only if the pull-back $f^*(a^k) \neq 0$.

We will frequently need to consider sweepouts of manifolds with boundary. In this case we consider the space of cycles relative to the boundary and all definitions above carry over to this setting.

The 1-sweepouts that we construct in this paper are nicer than an arbitrary 1-sweepout. After a small perturbation different cycles in it will not intersect each other and one can turn them into level sets of a function $f : M \rightarrow \mathbb{R}$. We summarize this in the following definition.

Definition 3.1. Let M be a Riemannian surface (possibly with boundary). Let $\text{int}(M)$ denote the interior of M . We say that z_t is a monotone sweepout if z_t is a sweepout of M and for each t cycle z_t can be represented by a finite collection of points and piecewise smooth simple closed curves, which satisfy the following condition. There exists a family of nested subsets $A_t \subset M$, $A_{t'} \subset A_t$ for all $t' < t$, such that z_t contains $\partial A_t \setminus \partial M$ and is contained in ∂A_t .

Since the cycles are nested and they can be glued into the fundamental class of M , it follows that A_0 is collection of points and A_1 is all of M . Below we use this property to concatenate sweepouts of two adjacent regions.

Lemma 3.2. *Let M be a Riemannian surface, possibly with boundary, and let γ be a relative 1-cycle composed of finitely many piecewise smooth closed curves that have not self-intersections or pairwise intersections and separate M into M_1 and M_2 . Suppose there exist monotone sweepouts of M_1 and M_2 of length at most L . Then there exists a monotone sweepout of M by cycles z_t , such that we can decompose z_t as a sum of 1-chains $z_t^1 + z_t^2$, where $l(z_t^1) \leq L + \epsilon$ and z_t^2 is contained in γ .*

Proof. By definition of a monotone sweepout for each $i = 1, 2$ there exists a family A_t^i of nested sets with $\text{int}(M_i) \cap \partial A_t^i \subseteq z_t^i \subseteq \partial A_t^i$. After a small perturbation that keeps A_t^i 's nested and increases lengths of cycles by at most ϵ we can assume that ∂A_t^i will intersect γ in a (possibly empty) finite collection of arcs and closed curves I_t^i with $I_t^i \subseteq I_{t'}^i$ if $t \leq t'$.

Define $A_t = A_{2t}^1$ for $t \in [0, \frac{1}{2}]$ and $A_t = A_1^1 \cup A_{\frac{t+1}{2}}^2$ for $t \in (\frac{1}{2}, 1]$. We define sweepout $z_t = \overline{\partial A_t \cap \text{int}(M)}$. For $t \leq \frac{1}{2}$ each cycle z_t can be decomposed into a chain that is contained in z_{2t}^1 and a chain $I_t^1 \subset \gamma$. For $t > \frac{1}{2}$ cycle z_t can be decomposed into a chain that is contained in $z_{\frac{t+1}{2}}^2$ and a chain $\gamma \setminus I_t^2$. \square

4. LENGTH-AREA METHOD

Given a closed Riemannian surface (M, h) by uniformization theorem there exists a conformal diffeomorphism $\phi : (M, h) \rightarrow (M, h_0)$ from (M, h) to a surface of constant curvature (M, h_0) . This conformal equivalence will play a key role in our construction of parametric sweepouts. For a subset $U \subset M$ we will write $\mu_0(U)$ to denote its area with respect to metric h_0 and $\mu(U)$ to denote its area with respect to h . Similarly, we will write $d(x, y)$, $B(x, r)$ and ∇ to denote distance function, closed metric ball of radius r about x , and gradient with respect to h and we let $d^0(x, y)$, $B^0(x, r)$, ∇^0 denote the corresponding quantities with respect to h_0 .

A key tool in this paper is an old technique sometimes called the length-area method (see, for example, [18]). It is based on the observation that the n 'th power of the absolute value of the gradient of a function (where n is the dimension of the space) times the volume element is a conformal invariant. Using this observation and coarea formula we can obtain the following lemma, which will be used throughout the paper.

Let $N_r^0(U)$ denote the set $\{x \in M : d^0(x, U) < r\}$ and $A_r^0(U) = N_r^0(U) \setminus U$.

Lemma 4.1. *Let U and V be open subsets of M with $U \subset V \subset M$. For any $r > 0$ there exists an open set U' with $U \subset U' \subset V \cap N_r^0(U)$, such that $l(\partial U' \cap V) \leq \frac{\sqrt{\mu_0(A_r^0(U) \cap V)}}{r} \sqrt{\mu(A_r^0(U) \cap V)}$.*

Proof. Let d_V^0 denote the distance function induced by the restriction of Riemannian metric h_0 to the open set V . Observe that for any two points x and y in V we have $d^0(x, y) \leq d_V^0(x, y)$. In particular, we have that $A_r^0(U, V) = \{x \in V : d_V^0(x, U) < r\} \setminus U \subset A_r^0(U) \cap V$. Define a function $f : V \setminus U \rightarrow \mathbb{R}$ by setting $f(x) = d_V^0(x, U)$. By Rademacher's theorem f is differentiable almost everywhere. By coarea formula we have

$$\int_{t=0}^r l(f^{-1}(t)) dt = \int_{A_r^0(U, V)} |\nabla f| d\mu$$

By Cauchy-Schwartz inequality this quantity can be bounded above by

$$\left(\int_{A_r^0(U, V)} |\nabla f|^2 d\mu \right)^{1/2} \mu(A_r^0(U, V))^{1/2}$$

We observe that $|\nabla f|^2 dV$ is a conformal invariant, so

$$\int_{A_r^0(U, V)} |\nabla f|^2 d\mu = \int_{A_r^0(U, V)} |\nabla^0 f|^2 d\mu_0 = \mu_0(A_r^0(U, V))$$

It follows that for some $l \in [0, r]$ the set $U' = f^{-1}([0, l]) \cup U$ has boundary length at most $\frac{\sqrt{\mu_0(A_r^0(U, V))}}{r} \sqrt{\mu(A_r^0(U, V))}$. □

5. TESSELLATIONS OF HYPERBOLIC SURFACES

We use the following tessellation of a Riemann surface due to Buser.

Proposition 5.1. *(Buser) Let Σ be a closed hyperbolic surface. There exists a tessellation of Σ into polygons $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ with the following properties:*

1. \mathcal{T}_1 is a collection of triangles with sidelengths between $\log(2)$ and $2\log(2)$ and areas between 0.19 and 0.55.
2. \mathcal{T}_2 is a collection of quadrilaterals (see figure 1) with three right angles and one angle $\phi > \pi/3$. The sidelengths satisfy the following relations: $a \leq \log(2)/2$, $\log(2)/2 \leq c \leq 0.45$ and $b \geq d \geq 0.57$. The area of each quadrilateral is between 0.26 and 0.34.
3. For each polygon $T \in \mathcal{T}$ the $1/2$ -neighbourhood of T is contained in at most 40 polygons of \mathcal{T} .

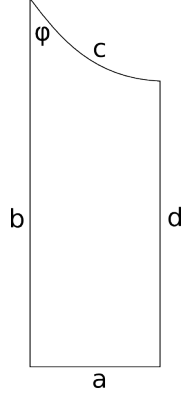


FIGURE 1. \mathcal{T}_2 consists of hyperbolic quadrilaterals with three right angles.

Proof. The construction of Buser ([4], p.116-121) relies on the thin-thick decomposition of Σ . Let β_1, \dots, β_k be the set of all simple closed geodesics of length $\leq \log(2)$ and let $w_i = \operatorname{arcsinh}(\frac{1}{\sinh(\frac{1}{2}|\beta_i|)}) > 1$. Then the tubular neighbourhood of β_i $C_i = \{p \in \Sigma | d(p, \beta_i) \leq w_i\}$ is isometric to the cylinder $[-w_i, w_i] \times S^1$ with the Riemannian metric $ds^2 = d\rho^2 + |\beta_i|^2 \cosh^2(\rho) dt^2$. Moreover, the cylinders C_i are disjoint.

In each collar C_i Buser defines two isometric annular regions, which he calls trigons. One boundary component of the trigon is the closed geodesic β_i and the other boundary component consists of two geodesic arcs of equal length. The endpoints of these geodesic arcs lie at a distance $w_i - \log(2)/2$ from β_i . Each trigon can be subdivided into four isometric quadrilateral as on Figure 1. These quadrilaterals have three right angle. A computation then yields the desired bounds on the sidelengths and the fourth angle. We define \mathcal{T}_2 to be the collection of all such quadrilaterals (eight in each collar).

In the remaining (thick) part of Σ the injectivity radius at a point x is bounded from below by $\min\{\log(2), d(x, V_2)\}$, where V denotes the set of vertices of quadrilaterals in \mathcal{T}_2 . Buser considers a maximal set of points at pairwise distances at least $\log(2)$. He then defines a geodesic triangulation of the thick part with this set as the set of vertices.

To prove the last assertion we observe that the worst case is when T is a triangle that is not adjacent to any of the quadrilaterals. As computed by Buser, all angles of the triangle are bounded below by 22.6° . It follows that $1/2$ -neighbourhood of T can be covered by less than 40 triangles. \square

6. SWEEPOUTS OF OPEN SUBSETS OF \mathbb{R}^2

Our proof of Proposition 7.2 relies on its Euclidean analogue. Namely, we need to know that for any open subset U of Euclidian plane there exists a sweepout of U by relative cycles of small length. This result was proved by Guth in [13] along with its high dimensional generalizations.

Theorem 6.1. (*Guth*) *Let $U \subset \mathbb{R}^2$ be a bounded open subset with piecewise smooth boundary. There exists a monotone sweepout of U by cycles of length $\leq 3\sqrt{\text{Area}(U)}$.*

Proof. We give an outline of the argument in [13]. The 2-dimensional case is significantly easier than the general inequality obtained by Guth for k -dimensional cycles sweeping out an open subset in \mathbb{R}^n .

At first one may hope that for some line $l \in \mathbb{R}^2$ the projection of U on l will have short fibers. However, there exist sets in \mathbb{R}^2 (known as Besicovitch sets) of arbitrarily small area such that any such projection will contain a fiber of length larger than 1.

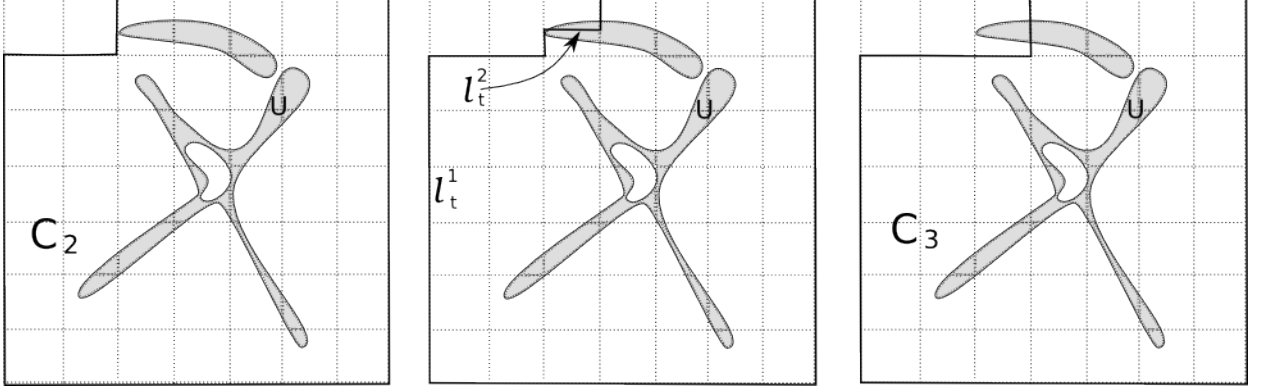
Instead of sweeping out U by parallel lines we will use cycles that are mostly contained in the 1-skeleton of a square grid. Scale U to have area 1. If we consider translates of the unit grid the total length of the intersection of the 1-skeleton (i.e. the union of the edges) with set U will have, on average, length equal to 2. (This can be seen as follows. First we translate the unit grid horizontally until the intersection of U with vertical lines of the grid has length 1; then we translate the grid vertically until the intersection of U with horizontal lines of the grid has length 1 giving us total length 2). Consider a large square $C_0 = [-N, N]^2$ that contains U and let $l_0 = \partial C_0$. Let $C_1 = C_0 \setminus [-N, -N+1] \times [N-1, N]$. Continue removing unit squares one by one (see Figure 2). This way we obtain N^2 connected unions of unit squares C_i with boundary in the 1-skeleton of the unit grid. Observe that one can homotop ∂C_i to ∂C_{i+1} via cycles that are contained in 1-skeleton except for a piece of length 1.

This gives rise to a family of nested open sets A_t , $A_{\frac{k}{4N^2}} = C_k$, and a homotopy $l_t = \partial A_t = l_t^1 + l_t^2$, where l_t^1 is contained in the unit grid and l_t^2 is either empty or an interval of length 1. Defining $z_t = \overline{\partial A_t \cap \text{int}(U)}$ we obtain a monotone sweepout with the desired length bound.

□

7. SWEEPOUTS OF SUBSETS COVERED BY A SMALL NUMBER OF POLYGONS

When the genus g of M is greater than or equal to 2 we scale (M, h_0) to have constant curvature -1 . By Gauss-Bonnet its volume satisfies $\mu_0(M) = 4\pi(g-1)$. By Proposition 5.1 there exists a tessellation \mathcal{T} of M into polygons.

FIGURE 2. Monotone sweepout of a subset of \mathbb{R}^2 .

When g is equal to 0 or 1 we scale the constant curvature space (sphere, projective plane, torus or a Klein bottle) so that it has volume 1. In this case we set \mathcal{T} to consist of only one element, the whole space M .

Lemma 7.1. *\mathcal{T} satisfies the following properties:*

- (1) $\#\mathcal{T} \leq \max\{67(g-1), 1\}$
- (2) Suppose $\{T_i\}_{i=1}^k \subset \mathcal{T}$, $k \leq 40$, and let $B^0(x, r)$ be any ball and let A denote the annulus $B^0(x, \frac{3r}{2}) \setminus B^0(x, r)$. There exists 42 balls $\{B^0(x_j, r)\}$, such that $A \cap \bigcup T_i \subset \bigcup B^0(x_j, r)$.

Proof. When genus $g \leq 1$ we have $\#\mathcal{T} = 1$. It is easy to show that an annulus in the plane $B(3/2) \setminus B(1) \subset \mathbb{R}^2$ can be covered by 5 discs of radius 1. A similar covering also works on the round sphere S^2 . We conclude that both properties hold when $g \leq 1$.

Suppose $g \geq 1$. The first property follows since areas of polygons in \mathcal{T} are bounded from below by 0.19.

To prove the second property we consider two cases. Suppose $B(x, r)$ is a ball with $r \geq 2$. We can cover every triangle in \mathcal{T} by a ball of radius $\log(2) < r$. The remaining points of $A \cap \bigcup T_i$ lie in quadrilaterals. A quadrilateral $T \in \mathcal{T}$ can be arbitrarily long, but it has to be narrow: by construction the distance from a point x on one of its long sides to the other long side is at most 0.45. We can assume that the length of the side d of T (see Fig. 1) is greater than 3 for otherwise we would have that T is contained in some ball of radius r .

Recall from Buser's construction of quadrilaterals that we described in the proof of Proposition 5.1 that T is contained in a hyperbolic collar along with other 7 isometric quadrilaterals. Four of them lie to one side of a closed geodesic β that cuts the collar in the middle and four of them lie to the other side of β . Let C_T denote the union of

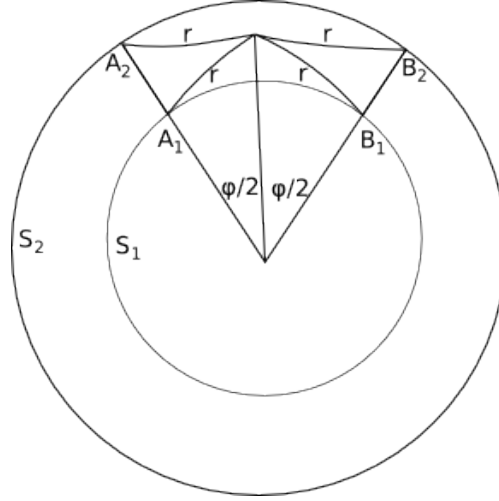


FIGURE 3. Covering annulus in hyperbolic plane

the 4 quadrilaterals that lie on the same side of β as T . We consider two possibilities. Suppose first that the center of the ball x does not lie in C_T . We observe that in this case $A \cap T$ is contained in a quadrilateral inside T that can be covered by one ball of radius r . Suppose $x \in C_T$. Then $A \cap T$ is contained in two subsets of T each of which can be covered by a ball of radius r . If other 3 quadrilaterals in C_T are not elements of \mathcal{T} it follows that we need at most 41 ball to cover $A \cap \bigcup T_i$. Notice also that all of $A \cap C_T$ can be covered by at most 4 balls of radius r . It follows then that the worst case is when exactly two quadrilaterals in C_T are elements of \mathcal{T} . Then we will need at most 42 balls.

Suppose $r \leq 2$. In this case we need only 21 balls $B^0(x_j, r)$ to cover A . This is illustrated on Figure 3. Consider two concentric circles S_1 and S_2 in the hyperbolic plane of radii r and $\frac{3}{2}r$ respectively. Suppose two geodesic rays emanating from x intersect circles S_1 and S_2 at A_1, B_1 and A_2, B_2 . For a correct value of the angle $\phi(r)$ between two geodesic rays we will have all four points lying on a circle of radius r . For $r \in [2, \infty)$ angle $\phi(r)$ is minimized when $r = 2$. We compute $\phi \geq 17.8^\circ$, so 21 discs will cover the annulus.

□

Proposition 7.2. *Let $U \subset M$ be an open subset with boundary and suppose there exists k sets $T_i \in \mathcal{T}$, $k \leq 40$, such that $U \subset \bigcup T_i$. Then there exists a monotone sweepout z_t of U , such that $l(z_t) \leq 489\sqrt{\mu(U)}$.*

We will inductively cut U into smaller pieces until the volume of each piece becomes so small that we can apply Proposition 7.3. We will then use Proposition 3.2 to concatenate these sweepouts into one sweepout.

Proposition 7.3. *For every $\epsilon > 0$ there exists a $\delta > 0$, such that for every open set $U \subset M$ with $\mu(U) < \delta^2$ there exists a monotone sweepout z_t of U of length $l(z_t) \leq \epsilon$.*

Proof. Choose $\delta > 0$ be smaller then the injectivity radius and suppose that it is small enough so that for every $x \in M$ and every $r \leq \delta$ the ball $B(x, r)$ with metric g restricted to it is 1.01-bilipschitz diffeomorphic to a disc of radius r in \mathbb{R}^2 .

We will show that there exists a monotone sweepout of U by cycles of length $\leq C \log(1/\delta^2)\delta$, where C is a constant that does not depend on δ (but depends on the volume of M). Note that we can make this quantity arbitrarily small by choosing sufficiently small δ .

Choose a maximal collection of disjoint balls in U of radius $\delta/6$. Let \mathcal{B} denote the collection of balls with the same centers and radius $\delta/2$. Observe that balls in \mathcal{B} cover U . Let k denote the number of balls in \mathcal{B} .

We claim that there exists a monotone sweepout z_t satisfying

$$(1) \quad l(z_t) \leq 500 \log(k+1)\delta$$

We prove equation (1) by induction on k . Suppose $k \leq 100$. By coarea inequality for each $B_i \in \mathcal{B}$ there exists a concentric ball $B'_i \supset B_i$ of radius r , $\delta/2 \leq r \leq \delta$, such that $l(\partial B'_i \cap U) \leq 2\delta$. By Theorem 6.1 there exists a monotone sweepout of $U \cap B'_i$ by cycles of length at most 4δ . Let B_j be a different ball in \mathcal{B} . As for B_i we can find a sweepout of $B'_j \cap (U \setminus B'_i)$ for some $B'_j \supset B_j$, such that B'_j has radius $\leq \delta$ and $l(\partial B'_j \cap (U \setminus B'_i)) \leq 2\delta$. By Lemma 3.2 there exists a monotone sweepout of $(B'_i \cup B'_j) \cap U$ by cycles of length $\leq 6\delta$. By repeating this step at most 100 times we obtain a monotone sweepout of U by cycles of length at most 204δ .

Assume the assertion holds for all U that can be covered by $< k$ balls of radius $\frac{1}{2}\delta$. Let k' be the smallest integer greater or equal to $k/100$ and let B denote the union of k' balls in \mathcal{B} . By coarea inequality there exists $r \leq \delta/2$, s.t. the boundary of the tubular neighbourhood $\partial(N_r(B) \cap U)$ has length at most 2δ . Set $U_1 = N_r(B) \cap U$. Since $N_r(B)$ is contained in the $\delta/2$ neighbourhood of B , it can be covered by at most $k/10 + 1$ balls of radius $\delta/2$. The set $U_2 = U \setminus N_r(B) \cap U$ can be covered by $\frac{99}{100}k$ balls in \mathcal{B} . By inductive assumption there exists a monotone sweepout of U_i , $i = 1, 2$, by cycles of length $\leq 500 \log(\frac{99}{100}k + 1)\delta$. By Lemma 3.2 there exists a sweepout of U by cycles of length at most $500 \log(\frac{99}{100}k)\delta + 2\delta < 500 \log(k)\delta$. This completes the proof of equation 1.

By definition of \mathcal{B} , balls with the same centers and $1/3$ of the radius are disjoint. In particular, the sum of their volumes is bounded above by $\mu(M)$. It follows that $k \leq 12 \frac{\mu(M)}{\delta^2}$. We conclude that $l(z_t) \leq C \log(1/\delta^2) \delta$ as desired. \square

We can now prove Proposition 7.2. Let $\epsilon < 0.001 \sqrt{\mu(U)}$ be a small number and choose $\delta(\epsilon) > 0$ as in Lemma 7.3. We will prove that for any subset $U' \subset U$ with piecewise smooth boundary there exists a monotone sweepout of U' by cycles of length $\leq 489 \sqrt{\mu(U')}$.

The proof proceeds by induction on $n = \log_{\frac{43}{44}}(\frac{\mu(U')}{\delta^2})$ and is reminiscent of arguments in [22]. When $\mu(U') \leq \delta^2$ we are done by Lemma 7.3. Assume the result to be true for all subsets of μ -volume $\leq (\frac{44}{43})^{n-1} \delta^2$ and consider $U' \subset U$ with $(\frac{44}{43})^{n-1} < \frac{\mu(U')}{\delta^2} \leq (\frac{44}{43})^n$.

Let r be the smallest radius, such that $\mu(B^0(x, r) \cap U') \geq \frac{\mu(U')}{44}$ for some $x \in M$. By Lemma 7.1 the intersection of the annulus $B^0(x, 3/2r) \setminus B^0(x, r)$ with U' can be covered by at most 42 balls $B^0(x_j, r)$. For each j we have $\mu(B^0(x_j, r) \cap U') \leq \frac{\mu(U')}{44}$ since $B^0(x, r)$ has maximal μ -volume for a ball of this radius. It follows that the total μ -volume of the set $A = (B^0(x, 3/2r) \setminus B^0(x, r)) \cap U'$ is bounded by $\frac{42}{44} \mu(U')$. By Lemma 4.1 we can find a relative cycle $\gamma \subset A$ of length $\leq 2 \frac{\sqrt{\mu_0(U')}}{r} \sqrt{\mu(U')}$ separating U' into two regions each having μ volume less or equal to $\frac{43}{44} \mu(U')$. Denote these two regions by U_1 and U_2 .

Now we derive a bound for the length of γ that does not depend on r . Since U' can be covered by at most 40 elements of \mathcal{T} its μ_0 -volume is bounded by $40 \times 0.55 = 22$ (recall that 0.55 is the maximal area of an element in \mathcal{T}). Hence, if $r > 1.68$ we obtain that $l(\gamma) \leq 5.58 \sqrt{\mu(U')}$.

On the other hand, suppose $r \leq 1.68$. In this case we can directly compute (using a formula for the area of a disc in a space of constant curvatures $-1, 0$ or 1) $\frac{2 \sqrt{\mu_0(A)}}{r} \leq 5.57$.

By inductive assumption both U_1 and U_2 admit a monotone sweepout with the desired length bound. By Lemma 3.2 there exists a monotone sweepout of U' by cycles of length $\leq 489 \sqrt{\frac{43}{44} \mu(U')} + 5.58 \sqrt{\mu(U')} + \epsilon \leq 489 \sqrt{\mu(U')}$.

This concludes the proof of Proposition 7.2.

8. GOOD COVERING OF M

Proposition 8.1. *Consider a surface M and let $U \subset M$ be an open subset with piecewise smooth boundary and suppose that it can be covered by m elements of \mathcal{T} . Let k be given. Then there exists a collection $\mathcal{U} = \{U_i\}$ of at most $m + \max\{m, 43k\}$*

sets, such that $\bigcup U_i$ covers U , $\mu(U_i \cap U) \leq \frac{\mu(U)}{k}$, each U_i is contained in at most 40 elements of \mathcal{T} and $l(\text{int}(U) \cap \bigcup \partial U_i) \leq (94.6\sqrt{m} + 36.6\sqrt{\max\{m, 43k\}})\sqrt{\mu(U)}$.

In the application of this Proposition to the proof of Theorem 1.1 we will take $U = M$.

Proof. Step 1. First we construct a covering of U by sets V_1, \dots, V_m , such that each V_i is contained in at most 40 polygons of \mathcal{T} , and the union of their boundaries satisfies a certain length bound. The μ -volume of each V_i , however, can be equal to anything between 0 and $\mu(U)$.

Let $\mathcal{T}' \subset \mathcal{T}$ be the set of m polygons that cover U and let $T_l \in \mathcal{T}'$ be such that $\mu(T_l \cap U) \geq \mu(T \cap U)$ for all $T \in \mathcal{T}'$. By Proposition 5.1 there are at most 39 polygons neighbouring T_l . The intersection of each of them with U has μ -volume less than or equal to $\mu(T_l \cap U)$. By the length-area argument Lemma 4.1 we can find set T' in the $1/2$ -neighbourhood of T_l , $T_l \subset T' \subset N_{1/2}(T_l)$, such that $l(\partial T' \cap U) \leq 2\sqrt{39} * 0.55\sqrt{39}\sqrt{\mu(T_l \cap U)} < 58\sqrt{\mu(T_l \cap U)}$. We set $T' = V_1$. We now apply the same construction to select a set $V_2 \subset U \setminus V_1$, such that V_2 can be covered by at most 40 polygons in \mathcal{T}' and $l(\partial V_2 \cap \text{int}(U \setminus V_1)) \leq 58\sqrt{\mu(V_2)}$. Each time we remove V_i the number of polygons necessary to cover the remaining part of U decreases by 1. Hence, we will be done after at most m steps. Since V_i have disjoint interiors we have $\sum \mu(V_i) = \mu(U)$. By Cauchy-Schwartz inequality the total length $l(\text{int}(U) \cap \bigcup \partial V_i) \leq 58 \sum \sqrt{\mu(V_i)} \leq 58\sqrt{m}\sqrt{\mu(U)}$.

Step 2. Let $N = \max\{m, 43k\}$. We subdivide each of V_i into a collection of subsets $\mathcal{U}_i = \{U_j^i\}$, such that each U_j^i has μ -volume at most $\frac{43\mu(U)}{N}$. Let k_i be the smallest integer larger than or equal to $N\mu(V_i)/\mu(U)$. Observe that $\sum k_i \leq N + m$.

If $k_i = 1$ we set $\mathcal{U}_i = \{V_i\}$. Suppose $k_i > 1$. Let $B^0(x, r)$ be a ball with the property that $\mu(B^0(x, r) \cap V_i) = \frac{\mu(U)}{N}$ and $\mu(B^0(y, r) \cap V_i) \leq \mu(B^0(x, r) \cap V_i)$ for any $y \in M$. Since V_i can be covered by at most 40 polygons, by Lemma 7.1 we have that $B^0(x, 3/2r) \cap V_i$ can be covered by at most 43 balls B^0 of radius r . It follows that μ -volume of $B^0(x, 3/2r) \cap V_i$ is at most $\frac{43\mu(U)}{N} \leq \frac{\mu(U)}{k}$.

As in the proof of Proposition 7.2 we can bound μ_0 -volume of the annulus $(B^0(x, 3/2r) \setminus B^0(x, r)) \cap V_i$. We separately consider the case when r is small ($r \leq 1.68$), and use comparison with the constant curvature space, and the case when r is large ($r > 1.68$) and use upper bound on the area of 40 polygons. By Lemma 4.1 we conclude that there exists a set $U_1^i \supset B^0(x, r) \cap V_i$ of volume at most $\frac{43\mu(U)}{N}$ and with $l(\text{int}(V_i) \cap \partial U_1^i) \leq 5.58\sqrt{\frac{43\mu(U)}{N}}$. Similarly, for each j we can find subsets U_j^i with disjoint interiors, μ -volume between $\frac{\mu(U)}{N}$ and $\frac{\mu(U)}{43N}$ and $l(\partial U_j^i \cap \text{int}(V_i \setminus (U_1^i \cup \dots \cup U_{j-1}^i))) \leq 5.58\sqrt{\frac{43\mu(U)}{N}}$. Observe that $\mathcal{U}_i = \{U_j^i\}$ has at most k_i elements.

We can now estimate the total length of the union of the boundaries $L = l(\text{int}(U) \cap \bigcup_{i,j} \partial U_j^i) \leq 58\sqrt{m}\sqrt{\mu(U)} + \sum k_i * 5.58\sqrt{\frac{43\mu(U)}{N}}$. The second term is bounded by $36.6(\frac{m}{\sqrt{N}} + \sqrt{N})\sqrt{\mu(U)}$. We conclude that the total length is bounded by $(94.6\sqrt{m} + 36.6\sqrt{N})\sqrt{\mu(U)}$. \square

9. PROOF OF THEOREM 1.1

Now we can prove Theorem 1.1. Let \mathcal{T} be a tessellation of M by (at most) $\max\{1, 67g\}$ polygons as in Lemma 7.1.

By Proposition 8.1 we can cover M by a collection of sets U_i each of μ -volume at most $\mu(M)/k$ and contained in at most 40 polygons of \mathcal{T} . The length of the union of the boundaries of sets U_i is bounded above by $(94.6\sqrt{67g} + 36.6\sqrt{\max\{67g, 43k\}})\sqrt{\mu(M)}$. Let N denote the number of sets in this covering.

First we construct a monotone 1-sweepout z_t of M . By Proposition 7.2 for each U_i there exists a monotone sweepout of U_i by cycles z_t^i of length at most $489\sqrt{\frac{\mu(M)}{k}}$. For $j/N \leq t \leq (j+1)/N$ we set $z_t = z_{Nt-j}^j + \sum_{i=1}^{j-1} z_1^i$. This defines a monotone sweepout of M with the property that each cycle can be written as a sum of chains $z_t = c_t^1 + c_t^2$, where c_t^1 has length at most $489\sqrt{\frac{\mu(M)}{k}}$ and c_t^2 is contained in $\bigcup \partial U_i$.

Consider truncated symmetric product $TP^k(S^1)$, i.e. all expressions of the form $\sum_{i=1}^k a_i t_i$, where $a_i \in \mathbb{Z}_2$ and $t_i \in S^1$. For any 1-sweepout z_t the family of cycles $\{\sum_{i=1}^k a_i z_{t_i}\}_{\sum_{i=1}^k a_i t_i \in TP^k(S^1)}$ is a k -sweepout of M (see [14], [7]).

We estimate the mass of each cycle

$$\begin{aligned} l\left(\sum_{i=1}^k a_i z_{t_i}\right) &\leq k \max_t \{l(c_t^1)\} + l\left(\bigcup \partial U_i\right) \\ &\leq (489\sqrt{k} + 94.6\sqrt{67g} + 36.6\sqrt{\max\{67g, 43k\}})\sqrt{\mu(M)} \end{aligned}$$

In particular, $l(\sum_{i=1}^k a_i z_{t_i}) \leq 1600 \max\{\sqrt{k}, \sqrt{g}\}\sqrt{\mu(M)}$. This concludes the proof of Theorem 1.1.

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